Prime Factorisation

If a natural number is expressed as the product of prime number (factors) then the factorisation of the number is called its prime factorisation.

i) 72= $2 \times 2 \times 2 \times 3 \times 3$

ii) 600= $2 \times 2 \times 2 \times 3 \times 5 \times 5$

iii) 9900= $2 \times 2 \times 3 \times 3 \times 5 \times 5 \times 11$

A) **Number of factors**

Let there be a composite number N and its prime factors be $a, b, c, d, \ldots$ etc. and $p, q, r, s, \ldots$ etc respectively $i.e.$, if N can be expressed as

$$N = a^p . b^q . c^r . d^s \ldots$$

Then the number of total divisors or factors of N is.

$$(p+1).(q+1)(r+1)(s+1) \ldots$$

Example: Find the number of factors of 24

24 is divisible by 1,2,3,4,6,8,12 and 24.

We see that there are total 8 factors namely

Lets use the method discussed above

24 = 8 x 3 = 2^3 x 3^1

So the number of factors = (3+1)(1+1) = 8 factors

EXAMPLE**:** Find the total number of factors of 360.

Solution: $360 = 2^3 \times 3^2 \times 5$

Since the powers are 3 , 2 and 1

$\therefore$ Number of factors $= (3+1) \times (2+1) \times (1+1) = 24$

EXAMPLE: Find the total number of factors of 540.

Solution: $540 = 2 \times 2 \times 3 \times 3 \times 5$

Or $540 = 2^2 \times 3^3 \times 5^1$

Therefore total number of factors of 540 is $(2+1)(3+1)(1+1) = 24$

EXAMPLE: The total number of divisors of 10500 except 1 and itself is:

Solutions:$10500 = 2 \times 2 \times 3 \times 5 \times 5 \times 5 \times 7$

Or $10500 = 2^2 \times 3^1 \times 5^3 \times 7^1$

$\therefore$ Total number of factors of 10500 is $(2+1)(1+1)(3+1)(1+1) = 48$

But we have to exclude 1 and 10500

So there are only $48 - 2 = 46$ factors of 10500 except 1 and 10500.

EXAMPLE: The total number of factors of 36 is:

Solution: $36 = 2^2 \times 3^2$

$\therefore$ Number of factors =$(2+1)(2+1) = 3 \times 3 = 9$

B) NUMBER OF WAYS OF EXPRESSING A COMPOSITE NUMBER AS A PRODUCT OF TWO FACTORS

Let us consider an example of small composite number say, 90
Then 90 $= 1 \times 90$
Or $= 2 \times 45$
Or $= 3 \times 30$
Or $= 5 \times 18$
Or $= 6 \times 15$
Or $= 9 \times 10$
So it is clear that the number of ways of expressing a composite no. as a product of two factors $= \frac{1}{2} \times$ the no. of total factors

Example: Find the number of ways of expressing 180 as a product of two factors.
Solution: 180 $= 2^2 \times 3^2 \times 5^1$
Number of factors $= (2+1)(2+1)(1+1) = 18$
Hence, there are total $\frac{18}{2} = 9$ ways in which 180 can be expressed as a product of two factors.

Perfect Squares: As you know when you express any perfect square number $'N'$ as a product of two factors as $\sqrt{N}$ x $\sqrt{N}$, and you also know that since in this case $\sqrt{N}$ appears two times but it is considered only once while calculating the no. of factors so we get an odd number as number of factors so we can not divide the odd number exactly by 2 as in the above formula. So if we have to consider these two same factors then we find the number of ways of expressing N as a product of two factors= $\frac{(Number\ of\ factors+1)}{2}$.

Non Perfect Squares: Again if it is asked that find the no. Of ways of expressing N as a product of two distinct factors then we do not consider 1 way ($i.e., N = \sqrt{N} \times \sqrt{N}$) then no. Of ways $= \frac{(Number\ of\ factors-1)}{2}$

Example: Find the number of ways of expressing 36 as a product of two factors.
Solution: 36 $= 2^2 \times 3^2$
Number of factors $= (2+1)(2+1) = 9$
Hence the no. Of ways of expressing 36 as product of two factors $= \frac{(9+1)}{2} = 5.$
As 36 $= 1 \times 36, 2 \times 18, 3 \times 12, 4 \times 9\ and\ 6 \times 6$

Example: In how many of ways can 576 be expressed as the product of two distinct factors?
Solution: 576 $= 2^6 \times 3^2$
∴ Total number of factors $= (6+1)(2+1) = 21$
So the number of ways of expressing 576 as a product of two distinct factors $= \frac{(21-1)}{2} = 10.$
Note since the word 'distinct' has been used therefore we do not include 576= 26×26.

C) NUMBER OF ODD FACTORS OF A GIVEN NUMBER

36= 1×36 also 36 $= 2^2 \times 3^2$

$= 2 \times 18$

$= 3 \times 12$

$= 4 \times 9$

$= 6 \times 6$

So, we can see the there are only 3 odd factors 1,3 and 9.

Once again we assume another number say 90

Then 90= 1×90 and also 90 $= 2 \times 3^2 \times 5$

$= 2 \times 45$

$= 3 \times 30$

$= 5 \times 18$

$= 6 \times 15$

$= 9 \times 10$

Thus there are only 6 odd factors namely 1,3,5,9,15,45.

Let N be the composite number and a, b, c, d.. be its prime factors and p, q, r, s be the indices (or powers) or a, b, c, d $i.e.$, if

$$N = a^p \times b^q \times c^r \times d^s \times x^l \times y^m$$

(where a, b, c, d… are the odd prime factors and x and y is the even prime factor.)

Then the total number of odd factors= $(p+1)(q+1)(r+1)$

Example: The number of odd factors (or divisors) of 24 is :

Solution: $24 = 2^3 \times 3^1$

Here 3 is the odd prime factor

So, total number of odd factors $= (1+1) = 2$

Example: The number of odd factors of 36 is…

Solution: $36 = 2^2 \times 3^2$

$\therefore$ number of odd factors $= (2+1)3$

Example: The number of odd factors of 90 is…

Solution: $90 = 2^1 \times 3^2 \times 5^1$

$\therefore$ total number of odd factors of $90 = (2+1)(1+1) = 6$

D) NUMBER OF EVEN FACTORS OF A COMPOSITE NUMBER

Number of even factors of a number = (Total number of factors of the given number— Total number of odd factors)

Thus the number of even factors of 24 = Number of factors of 24- Number of odd factors of $24 = 8 - 2 = 6$

Thus the number of even factors of 36 = Number of factors of 36- Number of odd factors of $36 = 9 - 3 = 6$

Thus the number of even factors of 90 = Number of factors of 90- Number of odd factors of $90 = 12 - 6 = 6$

Find the total number of even factors of 420.

Solution N= 420 = $2^2 \times 3^1 \times 7^1 \times 5^1$

Odd divisors will come only if we take zero power of 2 (since, any number multiplied by any power (> 1) of 2 will give us an even number)

So, odd divisors will come if we take 420 as $2^{\circ}x3^1x7^1x5^1$

So, number of odd divisors = (0 + 1) (1 + 1) (1 + 1) (1 + 1) = 8

So, total number of even divisors = total number of divisors - number of odd divisors = 24 - 8 = 16

Alternatively, we can also find out the number of even divisors of N = 420 directly (or, in general for any number).

420 = $2^2 \times 3^1 \times 7^1 \times 5^1$

To obtain the factors of 420 which are even, we will not consider 2°, since 2° = 1

So, the number of even divisors of 420 = (2) (1 + 1) (1 + 1)(1 + 1)=16

(We are not adding 1 in the power of 2, since we are not taking 2° here, i.e., we are not taking one power of 2.)

E) SUM OF FACTORS OF A GIVEN NUMBER

Let N be the composite number and a, b, c, d.. be its prime factors and p, q, r, s be the indices (or powers) or a, b, c, d $i.e.$, if N can be expressed as

Therefore, $N = a^p.b^q.c^r.d^s$ ...

Then the sum of all the divisors (or factors) of N

$$= \frac{(a^{p+1}-1)(b^{q+1}-1)(c^{r+1}-1)(d^{s+1}-1)}{(a-1)(b-1)(c-1)(d-1)}$$

EXAMPLE: Find the sum of factors of 24.

Solution: $24 = 2^3 \times 3^1$

$\therefore$ Sum of factors of $24 = \frac{(2^4-1)(3^2-1)}{(2-1)(3-1)}$

$$= \frac{15\times8}{1\times2} = 60$$

EXAMPLE: Find the sum of factors of 270.

Solution: $270 = 2 \times 3^3 \times 5$

$\therefore$ Sum of factors of 270$= \frac{(2^{1+1}-1)(3^{3+1}-1)(5^{1+1}-1)}{(2-1)(3-1)(5-1)}$

$$=\frac{3\times80\times24}{1\times2\times4} = 720$$

EXAMPLE: The sum of factors of 1520 except the unity is :

Solution: Since $1520 = 2^4 \times 5 \times 19$

$\therefore$ Sum of all the factors of $1520 = \frac{(2^5-1)(5^2-1)(19^2-1)}{(2-1)(5-1)(19-1)}$

$$= \frac{31\times24\times360}{1\times4\times18} = 3720$$

$\therefore$ The sum of the factors excluding unity = 3719

EXAMPLE: The sum of factors of 19600 is :

Solution: $19600 = 2^4 \times 5^2 \times 7^2$

$\therefore$ Sum of factors of 19600 $= \frac{(2^{4+1}-1)(5^{2+1}-1)(7^{2+1}-1)}{(2-1)(5-1)(7-1)}$

$= \frac{31 \times 124 \times 342}{1 \times 4 \times 6}$

$= 54777$

F) PRODUCT OF FACTORS

For example $24 = 1 \times 24$

Or, $24 = 2 \times 12$

Or, $24 = 3 \times 8$

Or, $24 = 4 \times 6$

Now, it is obvious from the above explanation that the product of factors of 24 is $(1 \times 24) \times (2 \times 12) \times (3 \times 8) \times (4 \times 6)$

$$= 24 \times 24 \times 24 \times 24 = (24)^4$$

Thus, the product of factors of composite number $N = N^{n/2}$, where n is the total number of factors of N.

Example: product of divisors of 7056 is :

Solution: $\therefore 7056 = 2^4 \times 3^2 \times 7^2$

$\therefore$ number of factors/divisors of 7056 $= (4+1)(2+1)(2+1) = 45$

$\therefore$ product of factors $= (7056)^{45/2} = (84)^{45}$

Example: product of factor of 360 is :

Solution: $360 = 2^3 \times 3^2 \times 5^1$

$\therefore$ number of factors of $= 360(3+1)(2+1)(1+1) = 24$

Thus the product of factors $= (360)^{24/2} = (360)^{12}$

Perfect Squares: As we know the number of divisors of any perfect number is an odd number so we take the square root of the perfect square to eliminate the $\frac{1}{2}$ From the power of the number as it can seen in the above example no. 1.

G) PAIR OF FACTORS CO-PRIME TO EACH OTHER

Let us see it for *N* = 18

Total number of factors of 12 = 6 (namely 1, 2, 3, 6, 9, 18)

Now. if we have to find out the set of factors of this number which are co-prime to each other, we can start with 1.

Number of factors which are co-prime to 1 = 5 (2, 3, 6, 9, 18)

Next in line is the number of factors which are co-prime to 2 = 2 (3,9)

So. the total number of set of factors for 12 which are co-prime to each other = 7

So. we can induce that if we have to find out the set of factors which are co-prime to each other for $N = a^p \times b^q$, will be equal to $[(p + 1)(q + 1) - 1 + pq]$

If there are three prime factors of the number, i.e., $N = a^p \times b^q \times c^r$, then set of co-prime factors can be given by [(p +1) *(q-* l)(r+ 1)- 1 *+ pq + qr + pr + 3pqr]*

Alternatively, we can find out the set of co-prime factors of this number by pairing it up first and then finding it out with the third factor.

Example 20 . Find the set of co-prime factors of the number N = 720.

Solution $720 = 2^4 \times 3^2 \times 5^1$

Using the formula for three prime factors *[{p +* 1) *(q* -l)(r+ 1)-1 *+ pq + qr + pr + 3pqr]*

We get, [(4 + 1) (2 + 1) (1 + 1) - 1 + 4.2 + 2.1 + 4.1 +3.4.2.1] = 67

Alternatively, let us find out first for $2^4 \times 3^2$= [(4 + 1) (2 + 1) - 1 + 4.2] = 22

Now $p^{22} \times 5^1$ will give us [(22 + 1) (1 + 1) - 1 + 22.1] = 67

Solved Examples

1) $N = 2^7 \times 3^s \times 5^6 \times 7^8$. How many factors of *N* are divisible by 50 but not by 100?
Solution**:** All the factors which are divisible by 50 but not divisible by 100 will have at least two powers of 5, and one power of 2.

And its format will be $2^1 \times 5^{2+y}$.

So, number of divisors =1x6x5x9 = 270

Alternatively, this is equal to (Number of factors divisible by 50) - (Number of factors divisible by 100).